

Statistical Models and Computing Methods, Problem Set 1

September 22, 2024

Due 10/07/2024

Problem 1.

(1) Show that $X \sim \mathcal{N}(0, 1)$ is the maximum entropy distribution such that $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$. (10 points)

(2) Generalize the result in (1) for the maximum entropy distribution given the first k moments, i.e., $\mathbb{E}X^i = m_i$, $i = 1, \dots, k$. (5 points)

Problem 2.

Let Y_1, \dots, Y_n be a set of independent random variables with the following pdfs

$$p(y_i|\theta_i) = \exp(y_i b(\theta_i) + c(\theta_i) + d(y_i)), \quad i = 1, \dots, n$$

Let $\mathbb{E}(Y_i) = \mu_i(\theta_i)$, $g(\mu_i) = x_i^T \beta$, where g is the link function and $\beta \in \mathbb{R}^d$ is the vector of model parameters.

(1) Denote $g(\mu_i)$ as η_i , and let s be the score function of β . Show that (15 points)

$$s_j = \sum_{i=1}^n \frac{(y_i - \mu_i) x_{ij}}{\text{Var}(Y_i)} \frac{\partial \mu_i}{\partial \eta_i}, \quad j = 1, \dots, d$$

(2) Let \mathcal{I} be the Fisher information matrix. Show that (5 points)

$$\mathcal{I}_{jk} = \mathbb{E}(s_j s_k) = \sum_{i=1}^n \frac{x_{ij} x_{ik}}{\text{Var}(Y_i)} \left(\frac{\partial \mu_i}{\partial \eta_i} \right)^2, \quad \forall 1 \leq j, k \leq d.$$

Problem 3.

Use the following code to generate covariate matrices X

```
1 import numpy as np
2 np.random.seed(1234)
3
4 n = 100
5 X = np.random.normal(size=(n, 2))
```

(1) Generate $n = 100$ observations Y following the logistic regression model with true parameter $\beta_0 = (-2, 1)$. (5 points)

(2) Find the MLE using the iteratively reweighted least square algorithm. (10 points)

(3) Repeat (1) and (2) for 100 instances. Compare the MLEs with the asymptotical distribution $\hat{\beta} \sim \mathcal{N}(\beta_0, \mathcal{I}^{-1}(\beta_0))$. Present your result with a scatter plot for MLEs with contours for the pdf of the asymptotical distribution. (10 points)

(4) Try the same for $n = 10000$. Does the asymptotical distribution provide a better fit to the MLEs? You can use the empirical covariance matrix of the MLEs for comparison. *(5 points)*

Problem 4.

Consider minimizing the following Beale's function

$$f(x, y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2$$

with an initial position at $(x_0, y_0) = (0.5, 2)$.

- (1) Find the global minima and show the contour plot over $[-4, 4] \times [-4, 4]$. *(5 points)*
- (2) Compare gradient descent, gradient descent with momentum and nesterov's accelerated gradient descent. Plot $f - f^*$ as a function of the number of iterations. *(15 points)*
- (3) Compare vanilla stochastic gradient descent with different adaptive stochastic gradient descent methods, including AdaGrad, RMSprop, and Adam. Plot $f - f^*$ as a function of the number of iterations. You can add some random noise (e.g., $\mathcal{N}(0, 0.01)$) to the exact gradient to form the stochastic gradient. *(15 points)*